

A STRENGTHENING OF WICKSTEAD'S THEOREM: ORDERED BANACH SPACES IN WHICH EVERY PRECOMPACT SET IS ORDER BOUNDED

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ABSTRACT. A theorem of Wickstead from 1975 characterizes the ordered Banach spaces with order bounded precompact sets in terms of a geometric property, “coadditivity”, relating the space’s order with its topology. We strengthen Wickstead’s Theorem by showing for an ordered Banach space to have all its precompact sets be order bounded, it is necessary and sufficient for the space to have all its null sequences be order bounded.

To establish our strengthening of Wickstead’s Theorem, we first prove an Open Mapping Theorem for cone-valued correspondences, which is then employed to prove a Klee-Andô type theorem for coadditivity (the classical Klee-Andô Theorem concerns another geometric property, namely “conormality”). By employing this Klee-Andô type theorem for coadditivity, we establish the equivalence of an ordered Banach space having the coadditivity property from Wickstead’s original result with the space having all its null sequences be order bounded.

Finally, for the purpose of illustration, we briefly investigate the natural order structures of the James space and the Tsirelson space. The James space is not a Banach lattice, but all its precompact sets are order bounded. The Tsirelson space is a Banach lattice, but not all its precompact sets are order bounded.

1. INTRODUCTION

Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ a collection of closed cones in X , indexed by a set Ω . Consider the following two geometric properties¹ $\{C_\omega\}_{\omega \in \Omega}$ could satisfy in X :

- (1) **Conormality:** There exists a constant $\alpha > 0$ such that, for every $x \in X$, there exists a decomposition $x = \sum_{\omega \in \Omega} c_\omega$ with $\sum_{\omega \in \Omega} \|c_\omega\| \leq \alpha \|x\|$ and $c_\omega \in C_\omega$ for all $\omega \in \Omega$.
- (2) **Coadditivity:** For some normed subspace Z of X^Ω , there exists a constant $\alpha > 0$ such that, for every $\xi \in Z$, there exists some $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ with $\|x\| \leq \alpha \|\xi\|$.

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¹Historically the properties “conormality” and “coadditivity” are respectively called “ α -generating” and “ α -directedness”. Our reason for deviating is in favor of the mnemonic device connecting “conormality” and “coadditivity” to their dual properties “normality” and “additivity” (standard terms which we do not define or need in this paper): Roughly, a space is normal (additive) if and only if its dual is conormal (coadditive), and vice versa. The interested reader is referred to [11].

The following two very simple examples are easily seen to be conormal and coadditive respectively, and also illustrates that these properties make sense even outside the realm of classical ordered Banach spaces.

Example 1.1. For \mathbb{R}^2 with the Euclidean norm with $\{e_1, e_2\}$ the standard basis for \mathbb{R}^2 , set $\Omega := \{e_1, e_2, -(e_1 + e_2)\} \subseteq \mathbb{R}^2$, and define the cones $\{C_x\}_{x \in \Omega}$ by setting $C_x := \{\lambda x \mid \lambda \geq 0\}$ for all $x \in \Omega$. The space \mathbb{R}^2 with $\{C_x\}_{x \in \Omega}$ is easily seen to be conormal.

Example 1.2. For \mathbb{R}^2 with the Euclidean norm define $C_1 := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$ and $C_2 := \{\alpha(1, 1) + \beta(1, -1) \in \mathbb{R}^2 \mid \alpha, \beta \geq 0\}$. The space \mathbb{R}^2 with $\{C_j\}_{j \in \{1, 2\}}$ is easily seen to be coadditive (with Z taken as the ℓ^2 -direct sum two copies of \mathbb{R}^2).

One result that motivates interest in such properties is the following, due to Wickstead. This result relates order boundedness of precompact sets in an ordered Banach space to the geometry of the cone of positive elements, namely having a certain coadditivity property:

Theorem 1.3 (Wickstead's Theorem [14, Theorem 1]). *Let X be a Banach space, ordered by a closed proper cone $C \subseteq X$. The following are equivalent:*

- (1) *There exists some $\alpha > 0$ such that, for every finite subset $F \subseteq X$, there exists some $y \in \bigcap_{x \in F} (x + C)$ satisfying $\|y\| \leq \alpha \max_{x \in F} \|x\|$.*
- (2) *Every precompact set in X is order bounded, i.e., for a precompact set K , there exist $x, y \in X$ such that $K \subseteq (x + C) \cap (y - C)$.*

In this paper we will strengthen Wickstead's Theorem by proving the following:

Theorem 1.4. *Let X be a Banach space, ordered by a closed proper cone. The following are equivalent:*

- (1) *All null sequences in X are order bounded.*
- (2) *All precompact sets in X are order bounded.*

Of course, that (2) implies (1) in Theorem 1.4 is trivial. We will establish the converse in this paper. This will be achieved through a “Klee-Andô Theorem for coadditivity”, which we will use to show that the order boundedness of all null sequences X is, in fact, equivalent to the coadditivity property (1) in Wickstead's Theorem, above.

We provide some background on the mentioned “Klee-Andô Theorem for coadditivity”.

The following result dates back to Andô's [3, Lemma 1], which was proven by employing an Open Mapping Theorem due to Klee [9, (3.2)]. The reader should recognize the conclusion of the following result as a conormality property.

Theorem 1.5 (Klee-Andô Theorem [3, Lemma 1], [9, (3.2)]). *Let X be a Banach space ordered by a closed cone $C \subseteq X$. If C is generating, i.e., $X = C - C$, then there exists a constant $\alpha > 0$ so that, for every $x \in X$, there exists a decomposition $x = a - b$ with $a, b \geq 0$ and $\|a\| + \|b\| \leq \alpha \|x\|$.*

Recently, a stronger version of the Klee-Andô Theorem, Corollary 5.6 below, was proven in [6]. Its proof employed a generalization of Banach's classical Open Mapping Theorem [6, Theorem 3.2] together with Michael's Selection Theorem [1, Theorem 17.66]. We note that Corollary 5.6 has wider applicability than ordered

Banach spaces (cf. Example 1.1) and has a distinctly more geometrical flavor than the order theoretic Theorem 1.5 (where we essentially restrict our attention to only two cones: C and $-C$).

Corollary 5.6 (Strong Klee-Andô Theorem for conormality [6, Theorem 4.1]). *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones in X . The following are equivalent:*

- (1) *For every $x \in X$, there exists a decomposition $x = \sum_{\omega \in \Omega} c_\omega$, with $c_\omega \in C_\omega$ for every $\omega \in \Omega$, and satisfying $\sum_{\omega \in \Omega} \|c_\omega\| < \infty$.*
- (2) *There exists an $\alpha > 0$ such that, for every $x \in X$, there exists a decomposition $x = \sum_{\omega \in \Omega} c_\omega$, with $c_\omega \in C_\omega$ for every $\omega \in \Omega$, and satisfying $\sum_{\omega \in \Omega} \|c_\omega\| \leq \alpha \|x\|$.*
- (3) *There exists an $\alpha > 0$ and, for every $\omega \in \Omega$, there exists a continuous positively homogeneous map $\delta_\omega : X \rightarrow C_\omega$ such that, for every $x \in X$, we have $x = \sum_{\omega \in \Omega} \delta_\omega(x)$ and $\sum_{\omega \in \Omega} \|\delta_\omega(x)\| \leq \alpha \|x\|$.*

In other words, the mere fact that one can decompose arbitrary elements of X as the limit of absolutely convergent series with terms chosen from the closed cones $\{C_\omega\}_{\omega \in \Omega}$, automatically implies that one can always choose such a decomposition in a bounded, continuous and positively homogeneous and manner. This is, of course, particularly useful when considering spaces of continuous functions taking values in an ordered Banach space (cf. [5, Corollary 2.8]).

Having now discussed the Strong Klee-Andô Theorem for conormality (Corollary 5.6, above), we point out that coadditivity and conormality are intimately related, in that both their statements satisfy the following general template:

For every structure A of a certain type, there exists some related structure B , and this B is bounded, in some sense, by A .

This relationship and the fact that the Klee-Andô Theorem for conormality (Corollary 5.6, above) was previously established in [6], raises the following question:

Question. *Does there exist a Klee-Andô type theorem for coadditivity? Roughly, if the intersection of certain translates of a collection of closed cones in a Banach space is always non-empty, can one always find an element in such an intersection in a bounded, continuous and positively homogeneous manner?*

We will answer this question positively in this paper by proving Corollary 5.5. A straightforward application of this result will then establish our strengthening of Wickstead's Theorem (Theorem 1.4).

Corollary 5.5 (Strong Klee-Andô Theorem for coadditivity). *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones in X . Let Z be either of the spaces $\mathbf{c}(\Omega, X)$ or $\ell^\infty(\Omega, X)$. Then the following are equivalent:*

- (1) *For every $\xi \in Z$, the intersection $\bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ is non-empty.*
- (2) *There exists an $\alpha > 0$ such that, for every $\xi \in Z$, there exists some $y \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ with $\|y\| \leq \alpha \|\xi\|_\infty$.*
- (3) *There exists an $\alpha > 0$ and a continuous positively homogeneous map $v : Z \rightarrow X$ such that, for every $\xi \in Z$, we have $v(\xi) \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ and $\|v(\xi)\| \leq \alpha \|\xi\|_\infty$.*

Our approach in proving Corollary 5.5 is similar in broad strokes to the proof of [6, Theorem 4.1], stated in this paper as Corollary 5.6. However, the Open Mapping Theorem [6, Theorem 3.2] employed in the proof of [6, Theorem 4.1] is not strong enough to establish the results leading up to Corollary 5.5. We further strengthen [6, Theorem 3.2] to yield Theorem 4.5, an Open Mapping Theorem for cone-valued correspondences, that is strong enough to establish our Strong Klee-Andô Theorems for both coadditivity and conormality (Corollaries 5.5 and 5.6 respectively in this paper):

Theorem 4.5 (Open Mapping Theorem for cone-valued correspondences). *Let C a complete metric cone (as defined in Section 4), Y a Banach space and $D \subseteq Y$ a closed cone. Let $T : C \rightarrow Y$ be a continuous additive positively homogeneous map. If the correspondence $\Psi : C \rightrightarrows Y$ defined by $\Psi(c) := Tc + D$ ($x \in C$) is surjective (in the sense that, for every $y \in Y$, there exists some $c \in C$ such that $y \in \Psi(c)$), then the image under Ψ of the open unit ball about zero in C is open in Y .*

We briefly describe the structure of the paper.

In Section 2 we provide some preliminary definitions and notation used throughout the current paper.

Sections 3 and 4 see the introduction of some terminology on correspondences (also known as multi-functions) and will prove some general results. Section 3 will introduce what we call “additive” and “ α -bounded” correspondences and will establish some general results that we will use in later sections. Section 4 sees the definition of metric cones and proof of one of our main results, an Open Mapping Theorem for cone-valued correspondences (Theorem 4.5).

We apply our results from the previous sections to prove our Strong Klee-Andô Theorems for conormality and coadditivity in Section 5. Although our subsequent results will rely only on our Klee-Andô Theorem for coadditivity, for the sake of completeness and illustration, we will also prove our Strong Klee-Andô Theorem for conormality (previously established in [6]).

A straightforward application of our Strong Klee-Andô Theorem for coadditivity will establish our claimed strengthening of Wickstead’s Theorem (Theorem 1.4) in Section 6.

Finally, in Section 7, we investigate the natural order structures of the James and Tsirelson spaces for the purpose of illustration. In particular we show that all precompact sets of the James space are order bounded, but that the same is not true for the Tsirelson space.

2. PRELIMINARY DEFINITIONS AND NOTATION

All vector spaces are assumed to be over the reals.

Let V be a vector space. A non-empty subset $C \subseteq V$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$ will be called a *cone*. If $C \cap -C = \{0\}$, then we will say C is a *proper cone*. If $V = C - C$, then we will say that C is *generating* in V . Translation invariant and positively homogeneous pre-orders (partial orders) on V are easily seen to be in bijection with cones (proper cones) in V , cf. [2, Section 1.1]. We will say V is *ordered by a cone* C by defining “ $v \leq w$ ” to mean $w \in v + C$ for $v, w \in V$.

For a normed space X , we will denote the unit sphere, closed unit ball and open unit ball of X respectively by \mathbf{S}_X , \mathbf{B}_X and \mathbb{B}_X .

Let Ω be an arbitrary index set, and let X be a Banach space.

- (1) For $1 \leq p \leq \infty$, by $\ell^p(\Omega, X)$ we will denote the usual ℓ^p -direct sum of $|\Omega|$ copies of X , with the norm on $\ell^p(\Omega, X)$ denoted by $\|\cdot\|_p$.
- (2) By $\mathbf{c}(\Omega, X)$ we will denote the closed subspace of $\ell^\infty(\Omega, X)$ of all elements $\xi \in \ell^\infty(\Omega, X)$ for which there exists some $x \in X$ such that, for every $\varepsilon > 0$, the set $\{\omega \in \Omega \mid \|\xi_\omega - x\| \geq \varepsilon\}$ is finite.
- (3) By $\mathbf{c}_0(\Omega, X)$ we will denote the closed subspace of $\ell^\infty(\Omega, X)$ of all elements $\xi \in \ell^\infty(\Omega, X)$ such that, for every $\varepsilon > 0$, the set $\{\omega \in \Omega \mid \|\xi_\omega\| \geq \varepsilon\}$ is finite.
- (4) As usual, we will denote $\mathbf{c}_0(\mathbb{N}, \mathbb{R})$ by c_0 , and the subspace of all finitely supported elements of c_0 will be denoted by c_{00} .

Let $\{C_\omega\}_{\omega \in \Omega}$ be an indexed collection of cones in X , and let Z be some vector subspace of X^Ω . The notation " $\bigoplus_{\omega \in \Omega} C_\omega \subseteq Z$ " will be used to denote the set $\{\xi \in Z \mid \forall \omega \in \Omega, \xi_\omega \in C_\omega\}$.

3. BOUNDED AND ADDITIVE CORRESPONDENCES

In this section our goal is to prove the general result, Proposition 3.3, which establishes the lower hemicontinuity of certain correspondences constructed from given correspondences having some extra algebraic structure.

We first introduce some terminology: Let A, B be sets. By a *correspondence* we mean a set valued map $\varphi : A \rightarrow 2^B$ and will use the notation $\varphi : A \rightrightarrows B$. If A and B are topological spaces, we will say that a correspondence $\varphi : A \rightrightarrows B$ is *lower hemicontinuous* if, for every $a \in A$ and every open set $U \subseteq B$ satisfying $\varphi(a) \cap U \neq \emptyset$, there exists some open set $V \ni a$ satisfying $\varphi(v) \cap U \neq \emptyset$ for all $v \in V$. By a *continuous selection* of φ we mean a continuous function $f : A \rightarrow B$ with $f(a) \in \varphi(a)$ for all $a \in A$.

Definition 3.1. Let X and Y be normed spaces and let $\varphi : X \rightrightarrows Y$ be a correspondence.

- (1) We will say φ is *additive* if, for $x, z \in X$, $\varphi(x) + \varphi(z) \subseteq \varphi(x + z)$.
- (2) For some $\alpha > 0$, we will say φ is α -*bounded* if, for every $x \in X$ and $\varepsilon > 0$, $\varphi(x) \cap (\alpha + \varepsilon) \|x\| \mathbf{B}_Y$ is non-empty.

We begin with the following elementary lemma, which is a crucial ingredient in the proof of Proposition 3.3.

Lemma 3.2. Let X be a normed space. Let $\alpha > 0$ and $G \subseteq X$ be a convex set such that, for every $\varepsilon > 0$, the set $G \cap (\alpha + \varepsilon) \mathbf{B}_X$ is non-empty. If, for an open set $U \subseteq X$ and some $\varepsilon_0 > 0$, the set $G \cap (\alpha + \varepsilon_0) \mathbf{B}_X \cap U$ is non-empty, then $G \cap (\alpha + \varepsilon_0) \mathbf{B}_X \cap U$ is also non-empty.

Proof. Let $U \subseteq X$ be open and $\varepsilon_0 > 0$ such that $G \cap (\alpha + \varepsilon_0) \mathbf{B}_X \cap U \neq \emptyset$. Let $x \in G \cap (\alpha + \varepsilon_0) \mathbf{B}_X \cap U$. If $x \in (\alpha + \varepsilon_0) \mathbf{B}_X$, then we are done. We therefore assume that $x \in (\alpha + \varepsilon_0) \mathbf{S}_X$. Let $y \in G \cap (\alpha + 2^{-1} \varepsilon_0) \mathbf{B}_X \neq \emptyset$. Then, for every $t \in (0, 1]$,

$$\begin{aligned}
 \|ty + (1 - t)x\| &\leq t\|y\| + (1 - t)\|x\| \\
 &\leq t(\alpha + 2^{-1} \varepsilon_0) + (1 - t)(\alpha + \varepsilon_0) \\
 &< t(\alpha + \varepsilon_0) + (1 - t)(\alpha + \varepsilon_0) \\
 &= (\alpha + \varepsilon_0).
 \end{aligned}$$

In other words, $ty + (1-t)x \in (\alpha + \varepsilon_0)\mathbb{B}_X$ for all $t \in (0, 1]$. Since $[0, 1] \ni t \mapsto ty + (1-t)x$ is continuous, there exists some $t_0 \in (0, 1]$ such that $t_0y + (1-t_0)x \in U$. Since G is convex, $t_0y + (1-t_0)x \in G$. We conclude $t_0y + (1-t_0)x \in G \cap (\alpha + \varepsilon_0)\mathbb{B}_X \cap U$. \square

Proposition 3.3. *Let X and Y be normed spaces and $\alpha > 0$. Let $\varphi : X \rightrightarrows Y$ be a convex-valued additive α -bounded correspondence. Then, for every $\varepsilon > 0$, the correspondence $\varphi_\varepsilon : \mathbf{S}_X \rightrightarrows Y$, defined by*

$$\varphi_\varepsilon(x) := \varphi(x) \cap (\alpha + \varepsilon)\mathbf{B}_Y \quad (x \in \mathbf{S}_X),$$

is non-empty- and convex-valued and lower hemicontinuous.

Proof. Since φ is α -bounded and convex-valued, that φ_ε is non-empty- and convex-valued for every $\varepsilon > 0$ is immediate.

We establish the lower hemicontinuity of φ_ε . Let $\varepsilon > 0$ and $x \in \mathbf{S}_X$ be arbitrary. Let $U \subseteq Y$ be open such that $\varphi_\varepsilon(x) \cap U = \varphi(x) \cap (\alpha + \varepsilon)\mathbf{B}_Y \cap U$ is non-empty. By Lemma 3.2, $\varphi(x) \cap (\alpha + \varepsilon)\mathbf{B}_Y \cap U$ is non-empty. Let $y \in \varphi(x) \cap (\alpha + \varepsilon)\mathbf{B}_Y \cap U$ be arbitrary, and let $r > 0$ be such that $y + r\mathbf{B}_Y \subseteq (\alpha + \varepsilon)\mathbf{B}_Y \cap U$. Now, let $z \in \mathbf{S}_X$ be such that $\|z - x\| < r(\alpha + \varepsilon)^{-1}$. Since φ is α -bounded, there exists some $v \in \varphi(z - x) \cap (\alpha + \varepsilon)\mathbf{B}_Y$. Then, $\|v\| \leq (\alpha + \varepsilon)\|z - x\| < r(\alpha + \varepsilon)^{-1}(\alpha + \varepsilon) = r$, so that $y + v \in y + r\mathbf{B}_Y \subseteq (\alpha + \varepsilon)\mathbf{B}_Y \cap U$, and, since φ is additive, we have $y + v \in \varphi(x + z - x) = \varphi(z)$. Therefore $y + v \in \varphi(z) \cap (\alpha + \varepsilon)\mathbf{B}_Y \cap U = \varphi_\varepsilon(z) \cap U$. Since z was chosen arbitrarily from $V := (x + r(\alpha + \varepsilon)^{-1}\mathbf{B}_X) \cap \mathbf{S}_X$, we conclude that φ_ε is lower hemicontinuous. \square

4. AN OPEN MAPPING THEOREM FOR CONE-VALUED CORRESPONDENCES

In this section we will prove one of our main results, namely an Open Mapping Theorem for cone-valued correspondences.

We begin with the following definitions and notation:

Definition 4.1. Let C be a set equipped with operations $+: C \times C \rightarrow C$ and $\cdot: \mathbb{R}_{\geq 0} \times C \rightarrow C$. The set C will be called an *abstract cone*, if there exists an element $0 \in C$ such that, for all $u, v, w \in C$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$, the following hold:

- (1) $u + 0 = u$
- (2) $(u + v) + w = u + (v + w)$
- (3) $u + v = v + u$
- (4) $u + v = u + w$ implies $v = w$
- (5) $1u = u$
- (6) $(\lambda\mu)u = \lambda(\mu u)$
- (7) $(\lambda + \mu)u = \lambda u + \mu u$
- (8) $\lambda(u + v) = \lambda u + \lambda v$.

Definition 4.2. Let C be an abstract cone and d a metric on C . The pair (C, d) will be called a *metric cone* if, for all $u, v, w \in C$ and $\lambda \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} d(0, \lambda u) &= \lambda d(0, u), \\ d(u + v, u + w) &\leq d(v, w). \end{aligned}$$

We introduce the notation $\llbracket x \rrbracket := d(0, x)$ for $x \in C$ and by \mathbb{B}_C we will denote the open unit ball about $0 \in C$, i.e., $\mathbb{B}_C := \{c \in C \mid \llbracket c \rrbracket < 1\}$.

Similarly to Banach spaces, if a metric cone is complete, then absolutely convergent series always converge.

Lemma 4.3. *Let (C, d) be a complete metric cone. If a sequence $\{c_i\} \subseteq C$ is such that $\sum_{i=1}^{\infty} \llbracket c_i \rrbracket$ converges, then the series $\sum_{i=1}^{\infty} c_i$ converges in C .*

Proof. Let C be a complete metric cone. Let $\{c_i\} \subseteq C$ be such that $\sum_{i=1}^{\infty} \llbracket c_i \rrbracket < \infty$. We claim that $\{\sum_{i=1}^n c_i\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ and assume $n \leq m$. Then

$$\begin{aligned} d\left(\sum_{i=1}^n c_i, \sum_{i=1}^m c_i\right) &\leq d\left(0, \sum_{i=n+1}^m c_i\right) \\ &\leq d(0, c_{n+1}) + d\left(c_{n+1}, \sum_{i=n+1}^m c_i\right) \\ &\leq \llbracket c_{n+1} \rrbracket + d\left(0, \sum_{i=n+2}^m c_i\right) \\ &\vdots \\ &\leq \sum_{i=n+1}^m \llbracket c_i \rrbracket. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \llbracket c_i \rrbracket < \infty$, by a standard argument, the tail $\sum_{i=n+1}^m \llbracket c_i \rrbracket$ can be made arbitrarily small for some M and $m, n \geq M$. We conclude that $\{\sum_{i=1}^n c_i\}_{n \in \mathbb{N}}$ is a Cauchy sequence, and hence converges in C . \square

To establish our Open Mapping Theorem, we will employ the Baire Category Theorem in the form of Zabreiko's Lemma:

Lemma 4.4 (Zabreiko's Lemma, [10, Lemma 1.6.3]). *Every countably subadditive seminorm on a Banach space is continuous.*

Finally, we will prove our main result of this section:

Theorem 4.5 (Open Mapping Theorem for cone-valued correspondences). *Let C be a complete metric cone, Y a Banach space and $D \subseteq Y$ a closed cone. Let $T : C \rightarrow Y$ be a continuous additive positively homogeneous map. If the correspondence $\Psi : C \rightarrow Y$ defined by $\Psi(c) := Tc + D$ ($c \in C$) is surjective, (in the sense that, for every $y \in Y$, there exists some $c \in C$ such that $y \in \Psi(c)$), then $\Psi(\mathbb{B}_C) \subseteq Y$ is open.*

Proof. Let Ψ , as defined, be surjective. We define the map $\rho : Y \rightarrow \mathbb{R}_{\geq 0}$ by

$$\rho(y) := \inf \{ \llbracket c \rrbracket \mid y \in \Psi(c) \}, \quad (y \in Y).$$

A straightforward calculation shows that ρ defines a seminorm on Y . We claim that ρ is countably subadditive.

Let $\{y_n\}_{n \in \mathbb{N}} \subseteq Y$ be such that the series $\sum_{n=1}^{\infty} y_n$ converges. If $\sum_{n=1}^{\infty} \rho(y_n) = \infty$ then $\rho(\sum_{n=1}^{\infty} y_n) \leq \sum_{n=1}^{\infty} \rho(y_n)$ holds trivially. We may therefore assume that $\sum_{n=1}^{\infty} \rho(y_n) < \infty$. Let $\varepsilon > 0$ be arbitrary and, for each $n \in \mathbb{N}$, let $c_n \in C$ be such that $y_n \in \Psi(c_n)$ and $\llbracket c_n \rrbracket < \rho(y_n) + 2^{-n}\varepsilon$. Then $\sum_{n=1}^{\infty} \llbracket c_n \rrbracket < \sum_{n=1}^{\infty} \rho(y_n) + \varepsilon$. Hence, by Lemma 4.3, the series $\sum_{n=1}^{\infty} c_n$ converges and $\llbracket \sum_{n=1}^{\infty} c_n \rrbracket < \sum_{n=1}^{\infty} \rho(y_n) + \varepsilon$.

For every $n \in \mathbb{N}$, we have $y_n \in \Psi(c_n) = Tc_n + D$, i.e., there exists some $d_n \in D$ such that $y_n - Tc_n = d_n$. Because T is continuous and additive, the series

$\sum_{n=1}^{\infty} (y_n - Tc_n)$ converges to $\sum_{n=1}^{\infty} y_n - T(\sum_{n=1}^{\infty} c_n)$. But $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} y_n - T(\sum_{n=1}^{\infty} c_n)$, and, in particular we note that the series $\sum_{n=1}^{\infty} d_n$ converges. Since D is a closed cone, the series $\sum_{n=1}^{\infty} d_n$ converges to a point in D . Therefore

$$\sum_{n=1}^{\infty} y_n = T\left(\sum_{n=1}^{\infty} c_n\right) + \sum_{n=1}^{\infty} d_n \in T\left(\sum_{n=1}^{\infty} c_n\right) + D,$$

and

$$\rho\left(\sum_{n=1}^{\infty} y_n\right) \leq \left\|\sum_{n=1}^{\infty} c_n\right\| \leq \sum_{n=1}^{\infty} \|c_n\| < \sum_{n=1}^{\infty} \rho(y_n) + \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, the claim that ρ is countably subadditive follows. We conclude that ρ is continuous by Zabreiko's Lemma (Lemma 4.4), and finally, that $\rho^{-1}([0, 1)) = \Psi(\mathbb{B}_C)$ is open. \square

5. STRONG KLEE-ANDÔ THEOREMS FOR CONORMALITY AND COADDITIVITY

We are now ready to establish our Strong Klee-Andô Theorems for conormality and coadditivity through an application of Theorem 4.5. Although our focus in this paper is on establishing a Klee-Andô Theorem for coadditivity (Corollary 5.5), for the sake of completeness and illustration we include a proof of a Klee-Andô Theorem conormality, Corollary 5.6 (also proven in [6]).

Theorem 5.1. *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ be an indexed collection of closed cones in X .*

- (1) *Let Z be either of the spaces $\mathbf{c}(\Omega, X)$ or $\ell^\infty(\Omega, X)$, and let the correspondence $\Upsilon : Z \rightarrow X$ be defined by*

$$\Upsilon(\xi) := \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega) \quad (\xi \in Z).$$

If Υ is non-empty-valued, then there exists some $\alpha > 0$ for which Υ is α -bounded.

- (2) *Let the correspondence $\Delta : X \rightarrow \ell^1(\Omega, X)$ be defined by*

$$\Delta(x) := \left\{ \xi \in \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^1(\Omega, X) \mid \sum_{\omega \in \Omega} \xi_\omega = x \right\} \quad (x \in X).$$

If Δ is non-empty-valued, then there exists some $\alpha > 0$ for which Δ is α -bounded.

Proof. We prove (1) in the case that $Z = \ell^\infty(\Omega, X)$. The case where $Z = \mathbf{c}(\Omega, X)$ follows similarly.

Let $\{C_\omega\}_{\omega \in \Omega}$ be such that, for every $\xi \in \ell^\infty(\Omega, X)$, $\bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega) \neq \emptyset$. Define $D := \bigoplus_{\omega \in \Omega} (-C_\omega) \subseteq \ell^\infty(\Omega, X)$ and $T : X \rightarrow \ell^\infty(\Omega, X)$ as $Tx := (\omega \mapsto x)$ for all $x \in X$ and $\omega \in \Omega$. The cone D is closed in $\ell^\infty(\Omega, X)$, and, since T is a linear isometry, it is clear that T is continuous, additive and positively homogeneous. We define $\Psi : X \rightarrow \ell^\infty(\Omega, X)$ by $\Psi(x) := Tx + D$ for all $x \in X$. It is easily seen that Ψ is surjective: Let $\xi \in \ell^\infty(\Omega, X)$ be arbitrary and choose $x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega) \neq \emptyset$, then $\xi \in Tx + D$. Now, by our Open Mapping Theorem (Theorem 4.5), $\Psi(\mathbb{B}_X)$ is open in $\ell^\infty(\Omega, X)$. Let $\beta > 0$ be such that $\beta \mathbb{B}_{\ell^\infty(\Omega, X)} \subseteq \Psi(\mathbb{B}_X)$. Then, for every $\xi \in \ell^\infty(\Omega, X)$ and $\varepsilon > 0$, there exists some $w \in \mathbb{B}_X$ such that $\beta \xi / (1 + \varepsilon \beta) \|\xi\|_\infty \in \Psi(w) = Tw + D$. Setting $x := (\beta^{-1} + \varepsilon) \|\xi\|_\infty w$ we obtain $\xi \in \Psi(x)$, implying

$x \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$, and $\|x\| \leq (\beta^{-1} + \varepsilon) \|\xi\|_\infty$. I.e., setting $\alpha := \beta^{-1}$, we have, for any $\xi \in \ell^\infty(\Omega, X)$ and $\varepsilon > 0$, that $\Upsilon(\xi) \cap (\alpha + \varepsilon) \|\xi\|_\infty \mathbf{B}_X \neq \emptyset$. We conclude that the correspondence Υ is α -bounded.

We prove (2). Let $\{C_\omega\}_{\omega \in \Omega}$ be such that for every $x \in X$, there exists some $\xi \in \ell^1(\Omega, X)$ such that $x = \sum_{\omega \in \Omega} \xi_\omega$. We define $C := \bigoplus_{\omega \in \Omega} C_\omega \subseteq \ell^1(\Omega, X)$ and $\Sigma : C \rightarrow X$ by $\Sigma\xi := \sum_{\omega \in \Omega} \xi_\omega$ for $\xi \in \ell^1(\Omega, X)$. Let $\Psi : \ell^1(\Omega, X) \rightrightarrows X$ be defined by $\Psi(\xi) := \Sigma\xi + \{0\}$. The cone C is a complete metric cone with the metric induced by the ℓ^1 -norm, and furthermore, Σ is surjective, continuous, additive and positively homogeneous. Therefore, by our Open Mapping Theorem (Theorem 4.5), $\Psi(\mathbb{B}_C)$ is an open set. Let $\beta > 0$ be such that $\beta\mathbb{B}_X \subseteq \Psi(\mathbb{B}_C)$. Then, for every $x \in X$ and $\varepsilon > 0$, there exists some $\eta \in \mathbb{B}_C$ such that $\beta x / (1 + \varepsilon\beta) \|x\| \in \Psi(\eta) = \sum_{\omega \in \Omega} \eta_\omega + \{0\}$. Setting $\xi := (\beta^{-1} + \varepsilon) \|x\| \eta$, we obtain $x \in \Psi(\xi)$, implying $x = \sum_{\omega \in \Omega} \xi_\omega$, and $\|\xi\|_1 \leq (\beta^{-1} + \varepsilon) \|x\|$. I.e., setting $\alpha := \beta^{-1}$, for any $x \in X$ and $\varepsilon > 0$, we obtain $\Delta(x) \cap (\alpha + \varepsilon) \|x\| \mathbf{B}_{\ell^1(\Omega, X)} \neq \emptyset$. We conclude that the correspondence Δ is α -bounded. \square

We now apply Proposition 3.3, to show that certain correspondences related to Υ and Δ are lower hemicontinuous.

Corollary 5.2. *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones.*

- (1) *Let Z be either of the spaces $\mathbf{c}(\Omega, X)$ or $\ell^\infty(\Omega, X)$. If $\Upsilon : Z \rightrightarrows X$, as defined in Theorem 5.1, is non-empty-valued, then there exists a constant $\alpha > 0$ such that, for every $\varepsilon > 0$, the correspondence $\Upsilon_\varepsilon : \mathbf{S}_Z \rightrightarrows X$, defined by*

$$\Upsilon_\varepsilon(\xi) := \Upsilon(\xi) \cap (\alpha + \varepsilon) \mathbf{B}_X \quad (\xi \in \mathbf{S}_Z),$$

is non-empty- closed- and convex-valued, and is lower hemicontinuous.

- (2) *If $\Delta : X \rightrightarrows \ell^1(\Omega, X)$, as defined in Theorem 5.1, is non-empty-valued, then there exists a constant $\alpha > 0$ such that, for every $\varepsilon > 0$, the correspondence $\Delta_\varepsilon : \mathbf{S}_X \rightrightarrows \ell^1(\Omega, X)$, defined by*

$$\Delta_\varepsilon(x) := \Delta(x) \cap (\alpha + \varepsilon) \mathbf{B}_{\ell^1(\Omega, X)} \quad (x \in \mathbf{S}_X),$$

is non-empty- closed- and convex-valued, and is lower hemicontinuous.

Proof. We prove (1). By Theorem 5.1(1), there exists some $\alpha > 0$ for which Υ is α -bounded. It is then clear that Υ_ε is then non-empty-, closed- and convex-valued for every $\varepsilon > 0$. It is easily seen that Υ is additive, so that, by Proposition 3.3, Υ_ε is lower hemicontinuous for every $\varepsilon > 0$.

We prove (2). By Theorem 5.1(2), there exists some $\alpha > 0$ for which Δ is α -bounded. Again, it is clear that Δ_ε is non-empty-, closed- and convex-valued for every $\varepsilon > 0$. That Δ is additive is easily seen, so that, by Proposition 3.3, Δ_ε is lower hemicontinuous for every $\varepsilon > 0$. \square

We will now apply Michael's Selection Theorem to obtain continuous selections of Δ and Υ .

Theorem 5.3 (Michael's Selection Theorem [1, Theorem 17.66]). *A lower hemicontinuous correspondence from a paracompact space into a Banach space with non-empty, closed and convex values admits a continuous selection.*

Corollary 5.4. *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones.*

- (1) *Let Z be either of the spaces $\mathbf{c}(\Omega, X)$ or $\ell^\infty(\Omega, X)$. If $\Upsilon : Z \rightrightarrows X$, as defined in Theorem 5.1, is non-empty-valued, then there exists a constant $\alpha > 0$ such that, for every $\varepsilon > 0$, there exists a continuous positively homogeneous selection $v : Z \rightarrow X$ of Υ such that $\|v(\xi)\| \leq (\alpha + \varepsilon) \|\xi\|_\infty$ for every $\xi \in Z$.*
- (2) *If $\Delta : X \rightrightarrows \ell^1(\Omega, X)$, as defined in Theorem 5.1, is non-empty-valued, then there exists a constant $\alpha > 0$ such that, for every $\varepsilon > 0$, there exists a continuous positively homogeneous selection $\delta : X \rightarrow \ell^1(\Omega, X)$ of Δ such that $\|\delta(x)\|_1 \leq (\alpha + \varepsilon) \|x\|$ for every $x \in X$.*

Proof. Before we begin, we note that all metric spaces are paracompact [13], so that, for any normed space N , its unit sphere \mathbf{S}_N is paracompact with the metric induced from the norm.

We prove (1). By Michael's Selection Theorem (Theorem 5.3), for every $\varepsilon > 0$, there exists a continuous selection $\underline{v} : \mathbf{S}_Z \rightarrow X$ of $\Upsilon_\varepsilon : \mathbf{S}_Z \rightrightarrows X$. We define v by

$$v(\xi) := \begin{cases} \|\xi\|_\infty \underline{v}\left(\frac{\xi}{\|\xi\|_\infty}\right) & \xi \neq 0 \\ 0 & \xi = 0 \end{cases} \quad (\xi \in Z),$$

which is easily seen to be continuous and positively homogeneous.

We prove (2). By Michael's Selection Theorem (Theorem 5.3), for every $\varepsilon > 0$, there exists a continuous selection $\underline{\delta} : \mathbf{S}_X \rightarrow \ell^1(\Omega, X)$ of $\Delta_\varepsilon : \mathbf{S}_X \rightrightarrows \ell^1(\Omega, X)$. By defining

$$\delta(x) := \begin{cases} \|x\| \underline{\delta}\left(\frac{x}{\|x\|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (x \in X),$$

which is easily seen to be continuous and positively homogeneous, we are done. \square

The following two results, our Strong Klee-Andô Theorems, now follow from Corollary 5.4. It is trivial that (3) \Rightarrow (2) \Rightarrow (1) in both results below. The implication (1) \Rightarrow (3) in both results below easily follow by fixing some $\varepsilon > 0$, say $\varepsilon := 1$, and applying Corollary 5.4.

Corollary 5.5 (Strong Klee-Andô Theorem for coadditivity). *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones in X . Let Z be either of the spaces $\mathbf{c}(\Omega, X)$ or $\ell^\infty(\Omega, X)$. Then the following are equivalent:*

- (1) *For every $\xi \in Z$, the intersection $\bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ is non-empty.*
- (2) *There exists an $\alpha > 0$ such that, for every $\xi \in Z$, there exists some $y \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ with $\|y\| \leq \alpha \|\xi\|_\infty$.*
- (3) *There exists an $\alpha > 0$ and a continuous positively homogeneous map $v : Z \rightarrow X$ such that, for every $\xi \in Z$, we have $v(\xi) \in \bigcap_{\omega \in \Omega} (\xi_\omega + C_\omega)$ and $\|v(\xi)\| \leq \alpha \|\xi\|_\infty$.*

Corollary 5.6 (Strong Klee-Andô Theorem for conormality). *Let X be a Banach space and $\{C_\omega\}_{\omega \in \Omega}$ an indexed collection of closed cones in X . The following are equivalent:*

- (1) *For every $x \in X$, there exists a decomposition $x = \sum_{\omega \in \Omega} c_\omega$, with $c_\omega \in C_\omega$ for every $\omega \in \Omega$, and satisfying $\sum_{\omega \in \Omega} \|c_\omega\| < \infty$.*

- (2) There exists an $\alpha > 0$ such that, for every $x \in X$, there exists a decomposition $x = \sum_{\omega \in \Omega} c_\omega$, with $c_\omega \in C_\omega$ for every $\omega \in \Omega$, and satisfying $\sum_{\omega \in \Omega} \|c_\omega\| \leq \alpha \|x\|$.
- (3) There exists an $\alpha > 0$ and, for every $\omega \in \Omega$, there exists a continuous positively homogeneous map $\delta_\omega : X \rightarrow C_\omega$ such that, for every $x \in X$, we have $x = \sum_{\omega \in \Omega} \delta_\omega(x)$ and $\sum_{\omega \in \Omega} \|\delta_\omega(x)\| \leq \alpha \|x\|$.

6. A STRENGTHENING OF WICKSTEAD'S THEOREM

In this section we prove our strengthening of Wickstead's Theorem (Theorem 1.4).

We begin with establishing the equivalence of the coadditivity properties (1) and (2) in the following result:

Proposition 6.1. *Let X be a Banach space, $C \subseteq X$ a closed cone, and Ω an infinite set. The following are equivalent:*

- (1) There exists some $\alpha > 0$ such that, for every finite set $F \subseteq X$, there exists some $y \in \bigcap_{x \in F} (x + C)$ satisfying $\|y\| \leq \alpha \max_{x \in F} \|x\|$.
- (2) There exists some $\alpha > 0$ such that, for every $\xi \in \mathbf{c}_0(\Omega, X)$, there exists some $y \in \bigcap_{\omega \in \Omega} (\xi_\omega + C)$ satisfying $\|y\| \leq \alpha \|\xi\|_\infty$.

Proof. We prove (1) implies (2).

Let $\alpha > 0$ be such that, for every finite set $F \subseteq X$, there exists some $y \in \bigcap_{x \in F} (x + C)$ satisfying $\|y\| \leq \alpha \max_{x \in F} \|x\|$. Let $\xi \in \mathbf{c}_0(\Omega, X)$ be arbitrary. Define the finite sets

$$\begin{aligned} J_1 &:= \{ \omega \in \Omega \mid 2^{-1} \|\xi\|_\infty \leq \|\xi_\omega\| \leq \|\xi\|_\infty \}, \\ J_n &:= \{ \omega \in \Omega \mid 2^{-n} \|\xi\|_\infty \leq \|\xi_\omega\| < 2^{-n+1} \|\xi\|_\infty \}, \quad 2 \leq n \in \mathbb{N}. \end{aligned}$$

It is clear that all these sets are disjoint. By our assumption, for every $n \in \mathbb{N}$ with $J_n \neq \emptyset$, there exists some $z_n \in \bigcap_{\omega \in J_n} (\xi_\omega + C) \cap C$ with $\|z_n\| \leq \alpha \max_{\omega \in J_n} \|\xi_\omega\| \leq 2^{-n+1} \alpha \|\xi\|_\infty$. For the indices $n \in \mathbb{N}$ with $J_n = \emptyset$, we set $z_n := 0$. We define $z := \sum_{j=1}^\infty z_j$, which is easily seen to converge absolutely, and note

$$\|z\| = \left\| \sum_{j=1}^\infty z_j \right\| \leq \sum_{j=1}^\infty \|z_j\| \leq \sum_{j=1}^\infty 2^{-j+1} \alpha \|\xi\|_\infty \leq 2\alpha \|\xi\|_\infty.$$

Since C is closed and $z_n \in C$ for all $n \in \mathbb{N}$, we have $z \in C$.

For every $\omega \in \Omega$, if $\xi_\omega = 0$ it is clear that $z \in \xi_\omega + C = C$. On the other hand, if $\xi_\omega \neq 0$, then there exists some $n \in \mathbb{N}$ with $\omega \in J_n$, so that $z = \sum_{j=1}^\infty z_j \in z_n + C \subseteq \xi_\omega + C$. We conclude that $z \in \bigcap_{\omega \in \Omega} (\xi_\omega + C)$ with $\|z\| \leq 2\alpha \|\xi\|_\infty$.

We prove (2) implies (1). Let $\alpha > 0$ such that, for every $\xi \in \mathbf{c}_0(\Omega, X)$, there exists some $y \in \bigcap_{\omega \in \Omega} (\xi_\omega + C)$ satisfying $\|y\| \leq \alpha \|\xi\|_\infty$. Let $F \subseteq X$ be a finite set. Since Ω is infinite, there exists an injection $i : F \rightarrow \Omega$. Define $\xi \in \mathbf{c}_0(\Omega, X)$ as

$$\xi_\omega := \begin{cases} x & \omega = i(x) \\ 0 & \omega \notin i(F) \end{cases} \quad (\omega \in \Omega).$$

Then there exists some $z \in \bigcap_{\omega \in \Omega} (\xi_\omega + C) \subseteq \bigcap_{x \in F} (x + C)$ satisfying $\|z\| \leq \alpha \|\xi\|_\infty = \alpha \max_{x \in F} \|x\|$. \square

Having now related the two coadditivity properties in the previous result, we can formulate the following strengthening of Wickstead's Theorem:

Corollary 6.2 (A strengthening of Wickstead's Theorem). *Let X be a Banach space and $C \subseteq X$ a closed proper cone. The following are equivalent:*

- (1) *For every $\xi \in \mathbf{c}(\mathbb{N}, X)$, the intersection $\bigcap_{n \in \mathbb{N}} (\xi_n + C)$ is non-empty.*
- (2) *For every $\xi \in \mathbf{c}_0(\mathbb{N}, X)$, the intersection $\bigcap_{n \in \mathbb{N}} (\xi_n + C)$ is non-empty.*
- (3) *There exists a constant $\alpha > 0$ such that, for every $\xi \in \mathbf{c}(\mathbb{N}, X)$, there exists some $y \in \bigcap_{n \in \mathbb{N}} (\xi_n + C)$ with $\|y\| \leq \alpha \|\xi\|_\infty$.*
- (4) *There exists a constant $\alpha > 0$ and a continuous positively homogeneous map $v : \mathbf{c}(\mathbb{N}, X) \rightarrow X$ such that, for every $\xi \in \mathbf{c}(\mathbb{N}, X)$, we have $v(\xi) \in \bigcap_{n \in \mathbb{N}} (\xi_n + C)$ and $\|v(\xi)\| \leq \alpha \|\xi\|_\infty$.*
- (5) *There exists a constant $\alpha > 0$ such that, for every finite set $F \subseteq X$, there exists some $y \in \bigcap_{x \in F} (x + C)$ with $\|y\| \leq \alpha \|x\|$.*
- (6) *Every precompact set in X is order bounded, i.e., for any precompact set $K \subseteq X$, there exist $x, y \in X$ satisfying $K \subseteq (x + C) \cap (y - C)$.*

Proof. That (1) implies (2) is trivial. That (2) implies (1) follows easily by translation. That (3) and (4) each imply (1) is trivial. That (1) implies (3) and (4) is a special case of Corollary 5.5 (with $\Omega := \mathbb{N}$ and $C_n := C$ for all $n \in \mathbb{N}$). The equivalence of (2) and (5) was proven in Proposition 6.1. Finally, the equivalence of (5) and (6) is Wickstead's original result (Theorem 1.3). \square

The equivalence of (2) and (6) in the previous result translates into the statement of Theorem 1.4 from the introduction.

7. EXAMPLES

In our final section we briefly analyze the natural order structures of the James and Tsirelson spaces. In particular, we will show that all precompact sets in the James space are order bounded, while this is not the case for the Tsirelson space.

7.1. The James space. We follow the definition of the James space from [8, Definition 6.37]. For $n \in \mathbb{N}$, let $\mathcal{P}_n := \{\{p_j\}_{j=1}^n \subseteq \mathbb{N} \mid 0 < p_1 < p_2 < \dots < p_n\}$. The James space J is defined as all elements $\xi \in c_0$ for which

$$\sup \left\{ \left(\sum_{j=1}^{n-1} (\xi_{p_{j+1}} - \xi_{p_j})^2 \right)^{1/2} \mid \begin{array}{l} 2 \leq n \in \mathbb{N}, \\ \{p_j\}_{j=1}^n \in \mathcal{P}_n \end{array} \right\} < \infty.$$

We define a norm $\|\cdot\|_J$ on J , by setting

$$\|\xi\|_J := \sup \left\{ \left(\sum_{j=1}^{n-1} (\xi_{p_{j+1}} - \xi_{p_j})^2 \right)^{1/2} \mid \begin{array}{l} 2 \leq n \in \mathbb{N}, \\ \{p_j\}_{j=1}^n \in \mathcal{P}_n \end{array} \right\} \quad (\xi \in J).$$

We endow J with its standard cone $J_+ := \{\xi \in J \mid \forall n \in \mathbb{N}, \xi_n \geq 0\}$.

Before proving Proposition 7.3, we state the following two lemmas. Their proofs are simple exercises and hence omitted.

Lemma 7.1. *Let the sequence $\{\xi^{(n)}\} \subseteq c_0$ converge to zero. Then, for every $\varepsilon > 0$, there exists some $N_\varepsilon \in \mathbb{N}$ such that $k > N_\varepsilon$ implies $|\xi_k^{(n)}| < \varepsilon$ for all $n \in \mathbb{N}$.*

Lemma 7.2. *If $a, b, c \in \mathbb{R}$ satisfy $0 \leq a \leq b \leq c$, then*

$$(a - b)^2 + (b - c)^2 \leq (a - c)^2.$$

We turn to the order structure of the James space.

Proposition 7.3. *For the James space J , ordered by its standard cone J_+ , the following are true:*

- (1) *The cone J_+ is closed, proper and generating in J .*
- (2) *For every $\alpha > 0$, there exist $\xi, \eta \in J$ such that $\pm \xi \leq \eta$ and $\|\xi\|_J > \alpha \|\eta\|_J$.*
- (3) *The James space is not a Banach lattice.*
- (4) *The inclusion map from the James space into c_0 is contractive.*
- (5) *If $\xi \in J_+$ is a decreasing sequence, i.e., ξ is such that $\xi_k \leq \xi_l$ for all $k, l \in \mathbb{N}$ with $k \geq l$, then $\|\xi\|_J = \|\xi\|_\infty$.*
- (6) *All null sequences in the James space are order bounded.*
- (7) *All compact sets in the James space are order bounded.*

Proof. We prove (1). That J_+ is a proper cone is immediate from its definition.

That J_+ is closed, follows from the fact that the standard basis $\{e_n\}$ inherited from c_0 is a Schauder basis for J , and that the coefficient functionals of a space with a Schauder basis are continuous [7, p. 33].

We show that J_+ is generating in J . Let $\xi \in J$, $2 \leq n \in \mathbb{N}$ and $\{p_j\}_{j=1}^n \in \mathcal{P}_n$ be arbitrary. By the reverse triangle inequality, we obtain $||\xi_{p_{j+1}}| - |\xi_{p_j}||^2 \leq |\xi_{p_{j+1}} - \xi_{p_j}|^2$ for every $j \in \{1, \dots, n-1\}$. Hence

$$\left(\sum_{j=1}^{n-1} (|\xi_{p_{j+1}}| - |\xi_{p_j}|)^2 \right)^{1/2} \leq \left(\sum_{j=1}^{n-1} (\xi_{p_{j+1}} - \xi_{p_j})^2 \right)^{1/2}.$$

Since $n \geq 2$ and $\{p_j\}_{j=1}^n \in \mathcal{P}_n$ were chosen arbitrarily, we obtain

$$\sup \left\{ \left(\sum_{j=1}^{n-1} (|\xi_{p_{j+1}}| - |\xi_{p_j}|)^2 \right)^{1/2} \mid 2 \leq n \in \mathbb{N}, \{p_j\}_{j=1}^n \in \mathcal{P}_n \right\} \leq \|\xi\|_J < \infty.$$

Therefore $|\xi| \in J$ (where $|\xi|_j := |\xi_j|$ for all $j \in \mathbb{N}$). Defining $\xi_\pm := \frac{1}{2}(|\xi| \pm \xi) \in J_+$, we see $\xi_+ - \xi_- = \frac{1}{2}(|\xi| + \xi) - \frac{1}{2}(|\xi| - \xi) = \xi$, and conclude that J_+ is generating in J .

We prove (2). For $n \in \mathbb{N}$, we define $\xi^{(n)} \in J$ by setting

$$\xi_k^{(n)} := \begin{cases} \sum_{j=1}^k (-1)^{j+1} j^{-1/2} & k \leq n \\ 0 & k > n \end{cases} \quad (k \in \mathbb{N}).$$

Then, for $n \in \mathbb{N}$,

$$\|\xi^{(n)}\|_J \geq \left(\sum_{j=1}^{n-1} (\xi_{j+1} - \xi_j)^2 \right)^{1/2}$$

$$= \left(\sum_{j=1}^{n-1} \frac{1}{j} \right)^{1/2} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Let $A \geq 0$ be such that $\left| \sum_{j=1}^k (-1)^{j+1} j^{-1/2} \right| < A$ for all $k \in \mathbb{N}$ (such an A exists, since the series $\sum_{j=1}^{\infty} (-1)^{j+1} j^{-1/2}$ converges by the alternating series test). For all $n \in \mathbb{N}$, define $\eta^{(n)} \in J$, by setting

$$\eta_k^{(n)} := \begin{cases} A & k \leq n \\ 0 & k > n \end{cases} \quad (k \in \mathbb{N}).$$

We note that $\|\eta^{(n)}\|_J = A$ and that $\pm \xi^{(n)} \leq \eta^{(n)}$ for all $n \in \mathbb{N}$. Then, for any $\alpha > 0$, let $n \in \mathbb{N}$ be such that $\|\xi^{(n)}\|_J > \alpha A$. Hence $\|\xi^{(n)}\|_J > \alpha \|\eta^{(n)}\|_J$.

We prove (3). If X is a Banach lattice then, for all $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \|y\|$. We proved in (2) that this is not true in J , and therefore J is not a Banach lattice.

We prove (4). Let $\xi \in J$ be arbitrary. Let $\varepsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that $j > N$ implies $|\xi_j| < \varepsilon$. For every $n \in \mathbb{N}$ and $m > \max\{n, N\}$,

$$\begin{aligned} |\xi_n| &< |\xi_n - \xi_m| + \varepsilon \\ &\leq \sup_{\{p_j\}_{j=1}^2 \in \mathcal{P}_2} \left(\sum_{j=1}^{n-1} (\xi_{p_{j+1}} - \xi_{p_j})^2 \right)^{1/2} + \varepsilon \\ &\leq \|\xi\|_J + \varepsilon. \end{aligned}$$

Taking the supremum over $n \in \mathbb{N}$ on both sides of the above inequality yields $\sup_{n \in \mathbb{N}} |\xi_n| \leq \|\xi\|_J + \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, the result follows, i.e., with $T : J \rightarrow c_0$ denoting the inclusion map, we have $\|T\xi\|_{\infty} \leq \|\xi\|_J$.

We prove (5). Let $\xi \in J_+$ be a decreasing sequence. By (4) of the current proposition, we have that $\|\xi\|_{\infty} \leq \|\xi\|_J$. For any $2 \leq n \in \mathbb{N}$ and $\{p_j\}_{j=1}^n \in \mathcal{P}_n$, by repeated application of Lemma 7.2, we notice that $\sum_{j=1}^{n-1} (\xi_{p_{j+1}} - \xi_{p_j})^2 \leq (\xi_{p_n} - \xi_{p_1})^2$. Therefore $\|\xi\|_J \leq \sup_{1 \leq l < k} |\xi_l - \xi_k|$. Furthermore, since $0 \leq \xi_k \leq \xi_l$ for $l \leq k$,

$$\begin{aligned} \|\xi\|_J &\leq \sup_{1 \leq l < k} |\xi_l - \xi_k| \\ &= \sup_{1 \leq l < k} \{\xi_l - \xi_k\} \\ &\leq \sup_{1 \leq l < k} \{\xi_l - \xi_k + \xi_k\} \\ &= \sup_{1 \leq l < k} \xi_l \\ &= \|\xi\|_{\infty}. \end{aligned}$$

We conclude that $\|\xi\|_{\infty} = \|\xi\|_J$.

We prove (6). Let the sequence $\{\xi^{(n)}\} \subseteq J$ converge to zero. If $\xi^{(n)} = 0$ for all $n \in \mathbb{N}$, then $0 \in J$ is an order bound for $\{\xi^{(n)}\}$. We therefore assume that $\sup_{n \in \mathbb{N}} \|\xi^{(n)}\|_{\infty} > 0$, and, by scaling, that $\sup_{n \in \mathbb{N}} \|\xi^{(n)}\|_{\infty} = 2$.

By (4) of the current proposition, $\{\xi^{(n)}\}$ also converges to zero in c_0 . By Lemma 7.1 there exists some $\kappa_1 \in \mathbb{N}$ such that, if $k \geq \kappa_1$ then $|\xi_k^{(n)}| < 1$ for

all $n \in \mathbb{N}$. We continue inductively in this fashion, by applying Lemma 7.1 for every $2 \leq l \in \mathbb{N}$, yielding some $\kappa_l \in \mathbb{N}$, chosen to satisfy $\kappa_l > \kappa_{l-1}$ and such that $k \geq \kappa_l$ implies $\left| \xi_k^{(n)} \right| < l^{-1}$ for all $n \in \mathbb{N}$.

We define $\eta \in c_0$ by setting

$$\eta_k := \begin{cases} \sup_{n \in \mathbb{N}} \|\xi^{(n)}\|_\infty & 1 \leq k < \kappa_1 \\ l^{-1} & \kappa_l \leq k < \kappa_{l+1}, \end{cases} \quad (k \in \mathbb{N}).$$

However, η is a decreasing sequence and therefore, by (5) of the current proposition, $\|\eta\|_J = \|\eta\|_\infty < \infty$. It is therefore clear that $\eta \in J_+$ and $\xi^{(n)} \leq \eta$ for all $n \in \mathbb{N}$.

Finally, (6) follows from (6) together with Corollary 6.2. \square

7.2. The Tsirelson space. We follow the Figiel-Johnson construction of the Tsirelson space from [4, Construction I.1]. Denote the set of all non-empty finite subsets of \mathbb{N} by $\mathcal{F}(\mathbb{N})$. For $E, F \in \mathcal{F}(\mathbb{N})$, by $E < F$ we mean $\max E < \min F$, and, for $k \in \mathbb{N}$, by $k < E$, $k \leq E$ and $k + E$ we mean $k < \min E$, $k \leq \min E$ and $\{k + x \mid x \in E\}$ respectively. For every $k \in \mathbb{N}$, we define $\mathcal{T}_k := \left\{ \{E_j\}_{j=1}^k \subseteq \mathcal{F}(\mathbb{N}) \mid k \leq E_1 < \dots < E_k \right\}$, and set $\mathcal{T} := \bigcup_{k \in \mathbb{N}} \mathcal{T}_k$. For $\xi \in c_{00}$ and $E \in \mathcal{F}(\mathbb{N})$, we define $E\xi \in c_{00}$ as

$$(E\xi)_k := \begin{cases} \xi_k & \text{if } k \in E \\ 0 & \text{if } k \notin E \end{cases} \quad (k \in \mathbb{N}).$$

For $\xi \in c_{00}$, we define $\|\xi\|^{(0)} := \max_{j \in \mathbb{N}} |\xi_j|$ and, for all $m \in \mathbb{N}$,

$$\|\xi\|^{(m)} := \max \left\{ \|\xi\|^{(m-1)}, \frac{1}{2} \max \left\{ \sum_{j=1}^k \|E_j \xi\|^{(m-1)} \mid \{E_j\}_{j=1}^k \in \mathcal{T} \right\} \right\}.$$

With $\|\xi\|_T := \lim_{m \rightarrow \infty} \|\xi\|^{(m)}$, the Tsirelson space T is defined as the completion of c_{00} with the norm $\|\cdot\|_T$. We define the Tsirelson space's standard cone T_+ as the closure in T of the standard cone in c_{00} : $\{\xi \in c_{00} \mid \forall n \in \mathbb{N}, \xi_n \geq 0\}$.

An elementary argument will establish the following:

Proposition 7.4. *The Tsirelson space, ordered by its standard cone T_+ , is a Banach lattice.*

Proof. A simple induction argument will show that c_{00} normed by $\|\cdot\|_T$ is a normed vector lattice (as defined in [12, Definition II.5.1]). By [12, Corollary 2, p. 84] the completion of a normed vector lattice is a Banach lattice. \square

Finally we show that the Tsirelson space does not enjoy the property of all its precompact sets being order bounded:

Proposition 7.5. *Let the Tsirelson space T be ordered by its standard cone T_+ . There exists a precompact set in T that is not order bounded.*

Proof. We establish this result by constructing a null sequence in T that is not order bounded.

For every $n \in \mathbb{N}$, let $k_n \in \mathbb{N}$ be the smallest integer satisfying $\sum_{m=n}^{k_n} m^{-1} \geq n$. We define $E_1 := \{1\}$ and $s_1 := 1$, and set

$$\begin{aligned} s_{n+1} &:= \max\{(\max E_n) + 1, k_{n+1}\}, \\ E_{n+1} &:= s_{n+1} + \{0, \dots, k_{n+1} - 1\}. \end{aligned}$$

For every $n \in \mathbb{N}$, we define $\eta^{(n)} \in c_{00}$ as

$$\eta_k^{(n)} := \begin{cases} 0 & \text{if } n \notin E_n \\ \frac{1}{m+n} & \text{if } s_n + m = k \in E_n, \end{cases} \quad (k \in \mathbb{N}).$$

We note that $\|\eta^{(n)}\|_T \geq \|\eta^{(n)}\|^{(1)} \geq \frac{1}{2} \sum_{m=n}^{k_n} \frac{1}{m} \geq \frac{1}{2}n$ for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we define $\xi^{(n)} \in c_{00}$ as

$$\xi_k^{(n)} := \begin{cases} 0 & \text{if } k \neq n \\ \max_{j \in \mathbb{N}} \eta_k^{(j)} & \text{if } k = n, \end{cases} \quad (k \in \mathbb{N}).$$

We note that $\|\xi^{(n)}\|_T = \|\xi^{(n)}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, i.e., $\{\xi^{(n)}\} \subseteq T$ is a null sequence. However, for every element $\zeta \in T$ satisfying $\zeta \geq \xi^{(n)}$ for all $n \in \mathbb{N}$, we must also have $\zeta \geq \eta^{(n)}$ for all $n \in \mathbb{N}$. Therefore $\|\zeta\|_T \geq \frac{1}{2}n$ for all $n \in \mathbb{N}$, which is impossible for $\zeta \in T$. \square

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